Series Representations and Rational Approximations for Hansen Coefficients

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New power series representations in the eccentricity and a related parameter are developed for the Hansen coefficients. The coefficients of the power series are easily obtained with simple recurrence relations generated from a partial differential equation. In many cases, the new series representations offer significant improvements in covergence for moderate to larger eccentricities. Rational approximations for the new representations are also investigated.

Introduction

THIS investigation focuses on new power series representations and rational function approximations for the Hansen coefficients that might offer computational advantages over the classical infinite series representations. The Hansen coefficients are the Fourier coefficients of the expansion

$$\left(\frac{r}{a}\right)^n \exp(isf) = \sum_{t=-\infty}^{\infty} X_t^{n,s}(e)\exp(it\ell) \tag{1}$$

where r is the magnitude of the radius vector, a and e the semimajor axis and eccentricity of the orbit, f and ℓ the true and mean anomaly, respectively, and i the imaginary unit. The coefficients are defined by the quadrature

$$X_t^{n,s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{r}{a}\right)^n \exp\, i(sf - t\ell)\,\mathrm{d}\ell \tag{2}$$

and possess the symmetry relation

$$X_t^{n,s} = X_{-t}^{n,-s} \tag{3}$$

These coefficients have played an important role in classical celestial mechanics and, more recently, in artificial satellite theory. The classical series representations for these coefficients were quite sufficient for the important classical problems that involve small values of the eccentricity. However, it is not uncommon to encounter artificial satellite orbits with eccentricities in the range $0.2 \le e \le 0.95$. For such large values of the eccentricity, the classical representation may not be the best representation, or in some cases even adequate, for computational purposes.

The Fourier series expansion in Eq. (1) and the classical series expansion for the Hansen coefficients converge[1-3] for all $e<1$. Covergence of the Hansen coefficients is readily deduced from its contour integral definition.[1,4,5] The radius of convergence contrasts with the radius of convergence $e<0.662$...for the power series expansion in e, with coefficients that depend on a fixed, but arbitrary, value of the mean anomaly for the left-hand side of Eq. (1), i.e.,

$$\left(\frac{r}{a}\right)^n \exp isf = \sum_{k=0}^{\infty} C_k(\ell)\frac{e^k}{k!} \tag{4}$$

This restricted radius of convergence is due to the radius of convergence for the power series expansion in e of Kepler's equation.[3] Thus, expansions of the form of Eq. (1) generally are to be favored over those of Eq. (4) and are essential for large enough values of the eccentricity.

The choice of power series representations for the Hansen coefficients over other infinite series forms is based on computational considerations. Specifically, the coefficients of the power series expansions for a given Hansen coefficient are rational constants. Thus, they can be computed once and stored for all later applications. The combination of constant coefficients and summing by Horner's rule[6] appears to result in near-minimal computational effort. The new series presented below closely parallel certain classical series; thus, this discussion commences with a summary of these classical representations.

Classical Series Representation

Two series representations found in the exhaustive work of Hansen[1] and reproduced by McClain[5] are as follows:

$$X_t^{n,s} = (1-\beta^2)^{2n+3}(1+\beta^2)^{-(n+1)}(-\beta)^{|t-s|} \times \sum_{i=0}^{\infty} L_{i+a}^{(n-t+1)}(\mu) L_{i+b}^{(n+t+1)}(\mu)\beta^{2i} \tag{5}$$

$$\mu = t(1-\beta^2)/(1+\beta^2) \tag{6}$$

and

$$X_t^{n,s} = (1+\beta^2)^{-(n+1)}(-\beta)^{|t-s|} \times \sum_{i=0}^{\infty} L_{i+a}^{(n-s-i-a-1)}(v) L_{i+b}^{(a+s-i-b-1)}(v)\beta^{2i} \tag{7}$$

$$v = t(1+\beta^2)^{-1} \tag{8}$$

where

$$\beta = e(1+\sqrt{1-e^2})^{-1} \tag{9}$$

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and the functions $L_m^{(j)}(x)$ are the generalized Laguerre polynomial defined[6] as

$$L_m^{(j)}(x)=\sum_{k=0}^{\infty}(-1)^k\binom{m+j}{m-k}\frac{x^k}{k!} \tag{10}$$

These polynomials are orthogonal and possesss the usual three-term recurrence relations. The indicies a and b are defined

$$a=[\,|t-s|+(t-s)\,]/2 \tag{11a}$$

$$b=[\,|t-s|-(t-s)\,]/2 \tag{11b}$$

Another classical representation of interest is the Newcomb-Poincaré power series in the square of the eccentricity.[7] It is expressed in Izsak's notation[8] by

$$X_t^{n,s}=\sum_{i=0}^{\infty}X_{i+a,i+b}^{n,s}e^{2i+a+b}=e^{|t-s|}\sum_{i=0}^{\infty}X_{i+a,i+b}^{n,s}e^{2i} \tag{12}$$

where a and b are defined by Eq. (11). The coefficients $X_{\rho,\sigma}^{n,s}$ are referred to as Newcomb operators and are polynomials in the indices n and s. For given n and s, these coefficients reduce to rational constants. They are easily computed by the recurrence relations of Von Zeipel.[9] Since the Von Zeipel technique provides the tool for generating the new power series representations, it is briefly reviewed here.

The Von Zeipel recurrence relations for the Newcomb operators are easily obtained from a partial differential equation given by Izsak et al.[8] and derived in McClain,[5]

$$(1-e^2)e\frac{\partial X^{n,s}}{\partial e}+(1-e^2)^{3/2}z\frac{\partial X^{n,s}}{\partial z}=\Big\{s[1-(1-e^2)^{3/2}]$$
$$+(s-n)\frac{e^2}{2}+(2s-n)ex+(s-n)\frac{e^2}{2}x^2\Big\}X^{n,s} \tag{13}$$

where

$$X^{n,s}=\left(\frac{r}{a}\right)^n\left(\frac{x}{z}\right)^s=\sum_{\rho=0}^{\infty}\sum_{\sigma=0}^{\infty}X_{\rho,\sigma}^{n,s}\,e^{\rho+\sigma}z^{\rho-\sigma} \tag{14}$$

and

$$x=\exp if \tag{15}$$

$$z=\exp i\ell \tag{16}$$

Making the substitution

$$x=z(x/z) \tag{17}$$

in Eq. (13) and developing the resulting equation into a power series in e yields the partial differential equation

$$2\left(e\frac{\partial}{\partial e}+z\frac{\partial}{\partial z}\right)X^{n,s}=2(2s-n)ezX^{n,s+1}$$
$$+(k-n)e^2z^2X^{n,s+2}+e^2\left[(4s-n)+2e\frac{\partial}{\partial e}+3z\frac{\partial}{\partial z}\right]X^{n,s}$$
$$-2\sum_{\tau\geq 2}\binom{3/2}{\tau}(-e^2)^{\tau}\left(s+z\frac{\partial}{\partial z}\right)X^{n,s} \tag{18}$$

Substituting Eq. (14) into Eq. (18) and collecting terms with like powers of the eccentricity yields the recurrence relation

$$4\rho X_{\rho,\sigma}^{n,s}=2(2s-n)X_{\rho-1,\sigma}^{n,s+1}+(s-n)X_{\rho-2,\sigma}^{n,s+2}$$
$$+(5\rho-\sigma-4+4s-n)X^{n,s}_{\rho-1,\sigma-1}$$
$$-2(\rho-\sigma+s)\sum_{\tau\geq 2}(-1)^{\tau}\binom{3/2}{\tau}X^{n,s}_{\rho-\tau,\sigma-\tau} \tag{19}$$

The subscripts in this relation are restricted to non-negative integers. Another recurrence relation can be obtained from Eq. (19) by interchanging the subscripts, changing s to $-s$, and using the symmetry relation

$$X_{\rho,\sigma}^{n,s}=X_{\sigma,\rho}^{n,-s} \tag{20}$$

which follows from the symmetry relation for the Hansen coefficients [Eq. (3)]. The result is

$$4\sigma X_{\rho,\sigma}^{n,s}=-2(2s+n)X_{\rho,\sigma-1}^{n,s-1}-(s+n)X_{\rho,\sigma-2}^{n,s-2}$$
$$-(\rho-5\sigma+4+4s+n)X^{n,s}_{\rho-1,\sigma-1}$$
$$+2(\rho-\sigma+s)\sum_{\tau\geq 2}(-1)^{\tau}\binom{3/2}{\tau}X^{n,s}_{\rho-\tau,\sigma-\tau} \tag{21}$$

Finally, a third recurrence relation can be obtained by summing Eqs. (19) and (21) to yield

$$4(\rho+\sigma)X_{\rho,\sigma}^{n,s}=2(2s-n)X_{\rho-1,\sigma}^{n,s+1}-2(2s+n)X_{\rho,\sigma-1}^{n,s-1}$$
$$+(s-n)X_{\rho-2,\sigma}^{n,s+2}-(s+n)X_{\rho,\sigma-2}^{n,s-2}+2(2\rho+2\sigma-4-n)X^{n,s}_{\rho-1,\sigma-1} \tag{22}$$

thus eliminating the summation over τ.

Initialization for the recurrence relation is provided by

$$X_{0,0}^{n,s}=1 \tag{23}$$

and the fact that quantities with negative subscripts are treated as identically zero. Consequently,

$$X_{1,0}^{n,s}=s-(n/2) \tag{24}$$

$$X_{0,1}^{n,s}=-s-(n/2) \tag{25}$$

etc., are easily obtained. Although the Newcomb operators are rational numbers, the problem of generating them can be reduced to integer arithmetic.[8]

New Power Series Representations

New power series are developed by imposing a suitable factorization on the generating function for the Hansen coefficients, which in turn yields a factorization of the coefficients themselves. The factorizations considered are selected because they occur naturally in the classical representations and because in some instances they isolate a high-order pole of the Hansen coefficient at $e=1$ for interior perturbations. Removing this pole from the series development improves convergence for large eccentricity. The new series representations are constructed using a generalization of the partial differential equation given by Izsak for generating the Von Zeipel recurrence relations for the Newcomb operators.

The power series developed in the following paragraphs parallel the three classical representations already given. Two power series in β corresponding to Hansen representations are presented first. Examination of the first Hansen representation [Eq. (5)] shows that for negative values of n the leading factor is large for large values of β (or e). (Negative values of n occur for interior perturbations, i.e., central body nonspherical gravity field or interior disturbing bodies.) To isolate this pole of order $2|n|-3$ at $e=1$, the factorization of the Hansen representation in Eq. (5) is retained. Only the Laguerre polynomial infinite series is to be recast as a power series in β^2. This can be accomplished by direct manipulation of the series. This approach, however, is tedious and leads to cumbersome expressions for the coefficients of the power series. Von Zeipel's technique yields simple recurrence relations analogous to those for the Newcomb operators. More generally, assume a factorization for the function $X^{n,s}$ defined in Eq. (14) of the form

$$X^{n,s}(\beta,z) = W_n(\beta^2)\,Y^{n,s}(\beta,z) \tag{26}$$

Furthermore, assume

$$\frac{dW_n}{d\beta} = f_n(\beta)\,W_n(\beta^2) \tag{27}$$

The analog of Von Zeipel's partial differential equation is then

$$\left[A\frac{\partial}{\partial\beta} + B\frac{\partial}{\partial z}\right] Y^{n,s} = [2sC + 2(s-n)D - Af_n(\beta)]\,Y^{n,s}$$

$$+ 2(2s-n)EzY^{n,s+1} + 2(s-n)Dz^2Y^{n,s+2} \tag{28}$$

where the coefficients A, B, C, D, and E are polynomials in β defined by

$$A = \beta(1-\beta^2)(1+\beta^2)^2 \tag{29a}$$

$$B = (1-\beta^2)^3 \tag{29b}$$

$$C = \beta^2(\beta^4+3) \tag{29c}$$

$$D = \beta^2(1+\beta^2) \tag{29d}$$

$$E = \beta(1+\beta^2) \tag{29e}$$

and where

$$Y^{n,s+j} = (x/z)^j\,Y^{n,s} \tag{30}$$

Q Series

For the first Hansen representation [Eq. (5)], we seek a new series of the form

$$X_t^{n,s} = (1-\beta^2)^{2n+3}(1+\beta^2)^{-(n+1)}$$

$$\times(-\beta)^{|t-s|}\sum_{i=0}^{\infty} Q^{n,s}_{i+a,i+b}\beta^{2i} \tag{31}$$

where a and b are defined in Eq. (11). In this case,

$$W_n(\beta^2) = (1-\beta^2)^{2n+3}(1+\beta^2)^{n+1} \tag{32}$$

Therefore

$$f_n(\beta) = -2\beta\left(\frac{2n+3}{1-\beta^2} + \frac{n+1}{1+\beta^2}\right) \tag{33}$$

In view of Eqs. (14) and (31), it follows that

$$Y^{n,s} = \sum_{\rho=0}^{\infty}\sum_{\sigma=0}^{\infty} Q^{n,s}_{\rho,\sigma}(-\beta)^{\rho+\sigma}z^{\rho-\sigma} \tag{34}$$

Substituting this form into Eq. (28), performing the indicated operations, and grouping like powers of β yields the recurrence relation

$$\rho Q^{n,s}_{\rho,\sigma} = (s-n)(Q^{n,s+2}_{\rho-2,\sigma} + Q^{n,s+2}_{\rho-3,\sigma-1})$$

$$+ (n-2s)(Q^{n,s+1}_{\rho-1,\sigma} + 2Q^{n,s+1}_{\rho-2,\sigma-1} + Q^{n,s+1}_{\rho-3,\sigma-2})$$

$$+ (4s+2n+\rho-2\sigma+5)Q^{n,s}_{\rho-1,\sigma-1}$$

$$+ (3n+s-\rho+2\sigma+4)Q^{n,s}_{\rho-2,\sigma-2}$$

$$+ (n+s+\rho-1)Q^{n,s}_{\rho-3,\sigma-3} \tag{35}$$

In view of the symmetry relation,

$$Q^{n,s}_{\rho,\sigma} = Q^{n,-s}_{\sigma,\rho} \tag{36}$$

which follows from Eq. (3), the second recurrence relation is obtained by setting s to $-s$ and interchanging ρ and σ to yield

$$\sigma Q^{n,s}_{\rho,\sigma} = -(s+n)(Q^{n,s-2}_{\rho,\sigma-2} + Q^{n,s-2}_{\rho-1,\sigma-3})$$

$$+ (2s+n)(Q^{n,s-1}_{\rho,\sigma-1} + Q^{n,s-1}_{\rho-1,\sigma-2} + Q^{n,s-1}_{\rho-2,\sigma-3})$$

$$+ [2(n-2s) + \sigma - 2\rho + 5]\,Q^{n,s}_{\rho-1,\sigma-1}$$

$$+ (3n-s-\sigma+2\rho+4)Q^{n,s}_{\rho-2,\sigma-2} + (n-s+\sigma-1)Q^{n,s}_{\rho-3,\sigma-3} \tag{37}$$

The recurrence relations are initialized with the values

$$Q^{n,s}_{0,0} = 1 \tag{38a}$$

$$Q^{n,s}_{\rho,\sigma} = 0 \quad \text{for } \rho<0 \text{ or } \sigma<0 \tag{38b}$$

These recurrence relations are of fixed length in contrast to the corresponding recursions for the Newcomb operators Eqs. (19) and (21). No apparent simplification is observed by summing these relations. Note, however, that summing the recurrence relations does eliminate the dependence on s of the coefficients for the back values along the diagonal. Both of the above recurrence formulas as well as their sum require nine back values, whereas the summed Newcomb operator recurrence [Eq. (22)] requires only five back values.

Z Series

The power series representation corresponding to the second Hansen series in Eq. (7) is obtained in the same manner with

$$X_t^{n,s} = (1+\beta^2)^{-(n+1)}(-\beta)^{|t-s|}\sum_{i=0}^{\infty} Z^{n,s}_{i+a,i+b}\beta^2 \tag{39}$$

$$W_n(\beta) = (1+\beta^2)^{-n-1} \tag{40}$$

and

$$Y^{n,s} = \sum_{\rho=0}^{\infty}\sum_{\sigma=0}^{\infty} Z^{n,s}_{\rho,\sigma}(-\beta)^{\rho+\sigma}z^{\rho-\sigma} \tag{41}$$

The recurrence relations for the coefficients $Z^{n,s}_{\rho,\sigma}$ are found to be

$$\rho Z^{n,s}_{\rho,\sigma} = (s-n)(Z^{n,s+2}_{\rho-2,\sigma} + Z^{n,s+2}_{\rho-3,\sigma-1})$$

$$+ (n-2s)(Z^{n,s+1}_{\rho-1,\sigma} + 2Z^{n,s+1}_{\rho-2,\sigma-1} + Z^{n,s+1}_{\rho-3,\sigma-2})$$

$$+ (4s+\rho-2\sigma+2)Z^{n,s}_{\rho-1,\sigma-1} + (s-n+2\sigma-\rho-2)Z^{n,s}_{\rho-2,\sigma-2}$$

$$+ (s-n+\rho-4)Z^{n,s}_{\rho-3,\sigma-3} \tag{42}$$

and

$$\sigma Z^{n,s}_{\rho,\sigma} = -(s+n)(Z^{n,s-2}_{\rho,\sigma-2} + Z^{n,s-2}_{\rho-1,\sigma-3})$$
$$+ (n+2s)(Z^{n,s-1}_{\rho,\sigma-1} + 2Z^{n,s-1}_{\rho-1,\sigma-2} + Z^{n,s-1}_{\rho-2,\sigma-3})$$
$$+ (-4s+\sigma-2\rho+2)Z^{n,s}_{\rho-1,\sigma-1} + (-s-n+2\rho-\sigma-2)$$
$$\times Z^{n,s}_{\rho-2,\sigma-2} + (-s-n+\sigma-4)Z^{n,s}_{\rho-3,\sigma-3} \tag{43}$$

The initialization for the recurrence relations is exactly the same as for the $Q^{n,s}_{\rho,\sigma}$ coefficients given in Eq. (38).

This power series representation was previously obtained by Baxter,[10] who formulated the coefficients $Z^{n,s}_{\rho,\sigma}$ as a sum of products of two polynomials depending on the indices $n,s,$ and t. The above algorithms offer a more direct procedure for obtaining these coefficients.

Numerical results indicate that the rate of convergence for the Q series is very sensitive to the value of β for very large eccentricities. This is also probably true for the Z series as well; and Baxter[10] has previously commented on the relatively slow convergence of the Z series. This convergence behavior can be explained by the fact that the Q coefficients have large magnitudes, $d\beta/de$ is a monotonically increasing function, and $d\beta/de \to \infty$ as $e \to 1$. Thus, a small change in the eccentricity causes a much larger change in β and therefore a significant increase in the number of terms required to maintain a given precision. To avoid this problem, a power series in the square of the eccentricity is considered.

Y Series

The final power series presented is a modification of the Newcomb-Poincaré power series in e^2. The quadrature defining the Hansen coefficients [Eq. (2)] can be expressed as a quadrature over the true anomaly,

$$X^{n,s}_t = (1-e^2)^{n+3/2}$$
$$\times \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\exp i(sf - t\ell)}{[1+e(\exp if + \exp -if)/2]^{n+2}} df \tag{44}$$

where the ratio r/a has been expressed as a function of the eccentricity and true anomaly. The power series representation for Eq. (44) is of the form

$$X^{n,s}_t = (1-e^2)^{n+3/2} e^{|t-s|} \sum_{i=0}^{\infty} Y^{n,s}_{i+a,i+b} e^{2i} \tag{45}$$

Also

$$W_n(e^2) = (1-e^2)^{n+3/2} \tag{46}$$

$$Y^{n,s} = \sum_{\rho=0}^{\infty}\sum_{\sigma=0}^{\infty} Y^{n,s}_{\rho,\sigma} e^{\rho+\sigma} Z^{\rho-\sigma} \tag{47}$$

where

$$X^{n,s}(e,z) = W_n(e^2)\, Y^{n,s}(e,z) \tag{48}$$

Substitution of these expressions into the Von Zeipel partial differential equation yields the recurrence relation

$$4\rho Y^{n,s}_{\rho,\sigma} = 2(2s-n) Y^{n,s+1}_{\rho-1,\sigma} + (s-n) Y^{n,s+2}_{\rho-2,\sigma}$$
$$+ (5\rho - \sigma + 3n + 4s + 2) Y^{n,s}_{\rho-1,\sigma-1}$$
$$+ (-s-\rho+\sigma) \sum_{\tau \geq 2} (-1)^{\tau} \binom{3/2}{\tau} Y^{n,s}_{\rho-\tau,\sigma-\tau} \tag{49}$$

and the symmetry condition yields

$$4\sigma Y^{n,s}_{\rho,\sigma} = -2(2s+n) Y^{n,s-1}_{\rho,\sigma-1} - (s+n) Y^{n,s-2}_{\rho,\sigma-2}$$
$$+ (5\sigma - \rho + 3n - 4s + 2) Y^{n,s}_{\rho-1,\sigma-1}$$
$$+ (s+\rho-\sigma) \sum_{\tau \geq 2} (-1)^{\tau} \binom{3/2}{\tau} Y^{n,s}_{\rho-\tau,\sigma-\tau} \tag{50}$$

Summing these recurrence relations yields

$$4(\rho+\sigma) Y^{n,s}_{\rho,\sigma} = 2(2s-n) Y^{n,s+1}_{\rho-1,\sigma} + (s-n) Y^{n,s+2}_{\rho-2,\sigma}$$
$$-2(2s+n) Y^{n,s-1}_{\rho,\sigma-1} - (s+n) Y^{n,s-2}_{\rho,\sigma-2}$$
$$+ (4(\rho+\sigma) + 6n + 4) Y^{n,s}_{\rho-1,\sigma-1} \tag{51}$$

thus eliminating the summation over τ. The initial values are identical to those given in Eqs. (23-25).

Rational Approximations

Rational approximation techniques are often used to speed the convergence of series approximations. The rational approximations developed in this investigation are obtained through continued fraction expansions and are related to the theory of Padé approximants.[11]

Let

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

be a power series expansion. Formally, it may be transformed into a continued fraction expansion

$$f(x) = \cfrac{C_0}{1 + \cfrac{C_1 x}{1 + \cfrac{C_2 x}{1 + \ldots}}} \tag{52}$$

where the coefficients C_j are obtained by the recurrent scheme

$$C_j = A^0_j \tag{53}$$

$$A^i_j = \frac{A^{i+1}_{j-2}}{A^0_{j-2}} - \frac{A^{i+1}_{j-1}}{A^0_{j-1}} \tag{54}$$

with initial conditions

$$A^i_0 = a_i \tag{55a}$$

$$A^i_1 = -a_{i+1}/a_0 \tag{55b}$$

This algorithm is well-defined for nonvanishing C_j. In particular, note that neither a_0 nor a_1 may be zero.

The convergents $F_n(x) = P_n(x)/Q_n(x)$ of the continued fraction expansion are easily computed using the recursions

$$P_{n+1} = P_n + C_{n+1} \times P_{n-1} \tag{56a}$$

$$Q_{n+1} = Q_n + C_{n+1} \times Q_{n-1} \tag{56b}$$

with

$$P_0 = C_0 \tag{57a}$$

$$P_{-1} = 0 \tag{57b}$$

$$Q_0 = 1 \tag{57c}$$

$$Q_{-1} = 1 \tag{57d}$$

Table 1 Dependence of convergence on n for $X_5^{n,0}$ at $e=0.7$, with 10^{-5} relative accuracy

n	N	N/P	Q	Q/P	Y	Y/P
−3	16	9	19	9	7	7
−4	19	9	20	9	8	7
−5	21	10	20	9	8	8
−6	24	12	21	9	9	9
−7	26	13	21	9	9	9
−8	28	13	21	9	9	9
−9	30	14	21	10	10	9
−10	32	16	21	10	10	9
−11	34	17	21	10	10	9
−12	36	18	21	12	11	9
−13	38	19	20	10	11	9
−14	40	18	20	12	11	9
−15	42	21	20	12	11	9
−16	44	22	19	10	11	9
−17	46	23	18	13	11	9
−18	48	26	17	13	11	9
−19	50	29	16	13	11	9
−20	51	27	15	13	10	9

Table 2 Dependence of convergence on n for $X_5^{n,0}$ at $e=0.8$, with 10^{-5} relative accuracy

n	N	N/P	Q	Q/P	Y	Y/P
−3	25	11	30	10	8	7
−4	30	12	31	10	9	9
−5	34	13	32	10	9	9
−6	38	15	32	10	9	9
−7	42	15	32	12	10	10
−8	46	17	33	12	10	10
−9	49	18	33	12	11	11
−10	53	20	33	13	11	10
−11	56	21	33	13	11	10
−12	59	24	32	13	12	10
−13	63	25	32	13	12	12
−14	66	27	32	13	12	11
−15	69	33	31	14	12	10
−16	72	37	31	13	12	10
−17	75	40	30	14	12	11
−18	78	43	29	15	12	10
−19	82	45	28	16	12	10
−20	84	55	26	17	12	10

Note that the nth convergent is obtained by processing the first n terms in the continued fraction expansion, which in turn are derived from the first n terms in the power series. Under favorable circumstances, the sequence $F_n(x)$ of convergents will converge to $f(x)$ more rapidly than the sequence of partial sums of the power series.

With such a possibility in mind, the continued fraction expansions were generated for the Newcomb-Poincaré series, as well as the Q and Y series. The numerical results suggest that the continued fraction developments converge.

As has been noted, the continued fraction algorithm requries that the first two power series coefficients a_0 and a_1 be nonzero. This requirement is not always met. When it is not, the tail of the power series is approximated with continued fractions. This ad hoc method has not yet failed in the numerical investigations.

Numerical Evaluation

To obtain insight into their convergence characteristics, the series representations given in Eqs. (12), (31), and (45) were evaluated on an Amdahl 470/V8 computer. Rational approximations to those series, obtained through continued fraction expansions, were also evaluated.

A very large number of coefficients $X_t^{n,s}$ was computed for values of e up to 0.9 to investigate general patterns of convergence behavior. The indices were varied over the ranges $-20 \le n < -3$, $|s| \le -n-1$, and $1 \le t \le 25$. In addition, a particular set of Hansen coefficients, important for satellites in 2:1 resonance at critical inclination and with eccentric orbits $e \simeq 0.7$, is considered in further detail.

A 10^{-5} relative difference criterion was used to establish convergence. This simple convergence test was used to initiate the study. More sophisticated criteria may be necessary in some cases.

Table 3 Dependence of convergence on t for $X_t^{-3,0}$ at $e=0.7$, with 10^{-5} relative accuracy

t	N	N/P	Q[a]	Q/P[a]	Y	Y/P
1	17	8	10	6	6	4
2	17	7	13	7	6	5
3	17	8	15	7	6	6
4	16	8	17	8	6	6
5	16	9	19	9	7	7
6	16	9	21	9	8	7
7	15	10	23	10	8	8
8	15	10	25	10	9	8
9	15	9	27	11	9	8
10	15	9	29	12	10	9
11	14	9	31	12	10	9
12	14	10	33	13	11	9
13	14	10	35	13	11	9
14	13	10	37	14	11	10
15	13	11	38	14	12	10
16	13	11	40	15	12	10
17	13	11		16	13	11
18	11	11		16	13	11
19	13	12		16	14	11
20	13	12		17	14	11
21	13	12		18	15	12
22	13	13			15	12
23	14	13			15	12
24	14	13			16	13
25	15	13			16	13

[a]Computations are discontinued for large t due to overflow.

Table 4 Dependence of convergence on t for $X_t^{-3,0}$ at $e=0.8$, with 10^{-5} relative accuracy

t	N	N/P	Q[a]	Q/P[a]	Y	Y/P
1	26	10	14	7	8	7
2	26	10	18	7	8	8
3	26	11	22	8	6	6
4	25	11	26	9	6	7
5	25	11	30	10	7	7
6	25	11	33	11	8	8
7	24	11	37	12	8	9
8	24	11	41	12	9	9
9	24	12	44	13	10	9
10	23	12	48	14	10	9
11	23	13	51	14	11	10
12	23	13		15	11	10
13	22	11		16	12	11
14	22	14		17	13	11
15	22	14		18	14	11
16	22	13		18	14	12
17	21	13		19	15	12
18	21	13		19	15	13
19	21	13			16	13
20	20	14			16	13
21	20	14			17	13
22	20	14			17	14
23	19	15			18	14
24	19	15			18	14
25	19	15			19	15

[a]Computations are discontinued for large t due to overflow.

Table 5 Dependence of convergence on s for $X_5^{-7,s}$ at $e = 0.7$, with 10^{-5} relative accuracy

s	N	N/P	Q	Q/P	Y	Y/P
−6	17	8	22	12	11	10
−5	26	11	18	9	7	7
−4	26	12	18	8	6	8
−3	26	12	19	8	7	8
−2	26	12	19	9	8	7
−1	26	13	20	9	8	8
0	26	13	21	9	9	9
1	26	13	22	10	10	10
2	26	13	24	12	11	11
3	26	14	26	13	12	13
4	27	16	29	14	13	15
5	28	17	32	17	15	17
6	15	13	34	17	15	17

Table 6 Convergence for the set of Hansen coefficients $X_t^{n,s}$ at $e = 0.7$, with relative accuracy 10^{-2}

n	s	t	N	N/P	Q	Q/P	Y	Y/P
−3	0	1	7	5	5	3	3	3
−3	2	1	2	2	7	5	3	5
−4	−1	1	9	5	4	3	2	3
−4	1	1	9	5	6	3	4	5
−5	0	2	10	7	7	5	4	4
−5	2	2	11	7	10	6	6	6
−6	−1	1	12	8	3	5	3	2
−6	1	1	13	8	3	5	3	4
−6	−1	2	12	7	6	4	4	3
−6	1	2	12	8	8	5	5	5

Table 7 Convergence for the set of Hansen coefficients $X_t^{n,s}$ at $e = 0.7$, with relative accuracy 10^{-5}

n	s	t	N	N/P	Q	Q/P	Y	Y/P
−3	0	1	17	8	10	6	6	4
−3	2	1	9	7	13	6	9	8
−4	−1	1	20	8	10	6	7	6
−4	1	1	20	9	11	6	7	8
−5	0	2	22	10	13	8	6	7
−5	2	2	23	12	16	9	7	9
−6	−1	1	25	10	9	7	6	7
−6	1	1	26	11	11	9	5	7
−6	−1	2	25	11	12	7	7	7
−6	1	2	25	10	14	9	7	9

All results presented in this study were obtained in IBM double-precision floating-point arithmetic. Questions may arise concerning the stability of the recurrences given in Eqs. (22), (35), (37), and (51) and of the associated continued fraction developments. Two points should be made. First, all methods used to compute the Hansen coefficients have consistently produced the same values. Second, the recursions governing the calculation of the power series coefficients are based on rational arithmetic and have integer analogs.[8] Thus, arbitrary accuracy is possible with multiple-precision integer arithmetic.

The convergence characteristics of the expansions vary with the indices n, s, and t and with the eccentricity. Several tables have been constructed to illustrate these variations. The nomenclature used in these tables is as follows:

N = Newcomb expansion
N/P = rational approximants of the Newcomb expansion
Q = Q expansion
Q/P = rational approximants of the Q expansion

Table 8 Convergence as a function of the eccentricity for the Hansen coefficient $X_1^{-3,0}$, with relative accuracy 10^{-2}

Expansion	0.65	0.68	0.7	0.71	0.74	0.77	0.8
N	6	6	7	7	8	9	10
N/P	4	5	5	5	5	5	5
Q	4	4	5	5	5	6	7
Q/P	3	3	3	3	3	4	4
Y	2	3	3	3	3	3	3
Y/P	3	3	3	3	3	3	3

Table 9 Convergence as a function of the eccentricity for the Hansen coefficient $X_1^{-3,0}$, with 10^{-5} relative accuracy

Expansion	0.65	0.68	0.7	0.71	0.74	0.77	0.8
N	14	16	17	18	20	23	26
N/P	7	7	8	8	8	9	10
Q	9	9	10	10	11	12	14
Q/P	5	6	6	6	6	6	7
Y	5	6	6	6	7	7	8
Y/P	4	4	7	7	7	7	7

Y = Y expansion
Y/P = rational approximants of the Y expansion

Tables 1–9 summarize the numerical results for a variety of tests. For the N, Q, and Y expansions, the tabular entries are the degree of e^2 (or of β^2) required to achieve convergence. The tabular entries under the columns designated N/P, Q/P, and Y/P represent the degree of the truncated series for N, Q, and Y, respectively, from which the rational approximates were obtained.

Tables 1 and 2 illustrate the dependence of convergence on the superscript n for each of the algorithms considered, at eccentricity values of 0.7 and 0.8, respectively. Notice that both Y and Y/P are quite robust, even as the eccentricity changes from 0.7 to 0.8. Both N/P and Q/P enhance the convergence properties of their respective power series, with Q/P exhibiting the more stable behavior. Notice that for very large n, the convergence of Q actually improves!

Tables 3 and 4 illustrate the variation in convergence as a function of the subscript t for the eccentricity values 0.7 and 0.8, respectively. Convergence slows dramatically for Q as t increases. Indeed, the Q expression coefficients become quite large for large t and computation with them requires special handling. The implementation used in this study was not designed to handle such special cases.

Once again Y and Y/P are the most robust expansions, requiring the fewest number of terms for convergence. But N/P exhibits nearly equivalent behavior. Even N is quite stable as t varies (but not as the eccentricity varies).

In Table 5, the manner in which convergence depends on the superscript s at $e = 0.7$ is illustrated. For each of Q, Q/P, Y, and Y/P, convergence seems to become more difficult as s increases, whereas for N and N/P the difficulty of convergence does not appear to increase. Even so, each of Y and Y/P exhibit the most rapid convergence, with Q/P almost as good. For large positive s, N/P also does well.

The numerical results on convergence thus far presented illustrate typical behavior. Based on the current evidence, Y and Y/P exhibit the most rapid convergence among the methods considered.

A particular set of Hansen coefficients, important for the Molniya class orbit are studied. For some applications, they may be needed with only limited accuracy. The computational behavior of this set is illustrated in Tables 6 and 7. Notice that to achieve a relative accuracy of 10^{-2}, a fourth-degree polynomial is required on average. To achieve 10^{-5} relative accuracy, a polynomial of degree six or seven is for the most part required in the Y expansion. In either case, the Y expansion affords the highest accuracy for the least cost.

Tables 8 and 9 illustrate the dependence of convergence on eccentricity for the various methods of calculation. Notice in particular that the Y expansion is insensitive to variations in e. A cubic polynomial in e^2 calculates the Hansen coefficient, with a relative accuracy of 10^{-2}, for all values of e between 0.65 and 0.8.

The Y-expansion algorithm has been implemented in the Draper Laboratory version of the Research and Development Goddard Trajectory Determination System (GTDS R&D) orbit propagator to model both the resonant tesseral and short-periodic tesseral dynamics for small and large eccentricity cases. This has resulted in a significant improvement over models based on the Newcomb-Poincaré series in computational efficiency and in memory reduction, while accurately modeling the tesseral perturbations.

Conclusions

This investigation is motivated by the need to rapidly and accurately compute the Hansen coefficients without using excessive storage. To maintain speed over variable, but large, values of eccentricity, computation with rapidly converging power series is desirable. New power series are developed by imposing a suitable factorization on the generating function for the Hansen coefficients, which in turn yields a factorization of the coefficients themselves. Numerical results indicate that the series exhibit significant variations in convergence behavior. Convergence is sensitive to the indices of the Hansen coefficient, as well as to the magnitude of the eccentricity. The Y expansion usually converges most rapidly and is the least sensitive to variations in the indices and in the eccentricity. Numerical results show that rational approximation techniques significantly improved the convergence of some series. Only marginal improvement is observed for the Y expansion, but it converges so well, when compared to the other series and to their rational approximations, that it is clearly the expansion of choice. Other power series expansions for the Hansen coefficients may be developed using the methods described in this paper.

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